Equality holds exactly when $B = \dfrac{\pi}{3}$. Here is a proof. Using the formulas

$$\sum_{\text{cyclic}} \cos^2 A = 1 - 2\cos A \cos B \cos C$$

and

$$\sum_{\text{cyclic}} (\cos A \cos B - \sin A \sin B) = \sum_{\text{cyclic}} \cos(A+B) = -\sum_{\text{cyclic}} \cos A,$$

we get :

$$\begin{aligned}
&3\left(\sum_{\text{cyclic}} \cos A\right)^2 - \left(\sum_{\text{cyclic}} \sin A\right)^2 \\
&= 3\sum_{\text{cyclic}} \cos^2 A + 6\sum_{\text{cyclic}} \cos A\cos B - \sum_{\text{cyclic}} \sin^2 A - 2\sum_{\text{cyclic}} \sin A \sin B \\
&= 3\sum_{\text{cyclic}} \cos^2 A - \sum_{\text{cyclic}} (1-\cos^2 A) + 6\sum_{\text{cyclic}} \cos A\cos B - 2\sum_{\text{cyclic}} \sin A\sin B \\
&= 4\sum_{\text{cyclic}} \cos^2 A - 3 + 4\sum_{\text{cyclic}} \cos A\cos B + 2\left(\sum_{\text{cyclic}} \cos A\cos B - \sum_{\text{cyclic}} \sin A \sin B\right) \\
&= 4(1 - 2\cos A\cos B\cos C) - 3 + 4\sum_{\text{cyclic}} \cos A\cos B - 2\sum_{\text{cyclic}} \cos A \\
&= (1-2\cos A)(1-2\cos B)(1-2\cos C).
\end{aligned}$$

Since $A \le \frac{\pi}{3}$ and $C \ge \frac{\pi}{3}$, it follows that $\cos A \ge \frac{1}{2}$ and $\cos C \le \frac{1}{2}$, hence

$$\sqrt{3}\sum_{\text{cyclic}} \cos A - \sum_{\text{cyclic}} \sin A$$

has the same sign as $2\cos B - 1$, and we are done.

3774. [2012 : 334, 336] *Proposed by P. H. O. Pantoja.*

Let a, b, c be positive real numbers. Prove that

$$\frac{c(a^2+b^2)}{\sqrt{a^3+b^3}} + \frac{a(b^2+c^2)}{\sqrt{b^3+c^3}} + \frac{b(c^2+a^2)}{\sqrt{c^3+a^3}} \le \frac{3}{4}(a^2+b^2+c^2+a+b+c)\,.$$

Solved by A. Alt ; G. Apostolopoulos ; C. Curtis ; O. Geupel ; O. Kouba ; D. Koukakis ; S. Malikić ; P. Perfetti ; D. Smith ; T. Zvonaru ; and the proposer. We present the solution by Arkady Alt.

We will prove the following stronger inequality :

$$\sum_{\text{cyclic}} \frac{c(a^2+b^2)}{\sqrt{a^3+b^3}} \le \frac{\sqrt{2}}{2}(a^2+b^2+c^2+a+b+c). \tag{1}$$

Note first that since $(a+b)(a^3+b^3)-(a^2+b^2)^2 = ab(a-b)^2 \ge 0$, we have

$$\frac{a^2+b^2}{\sqrt{a^3+b^3}} \le \sqrt{a+b}.$$

Hence

$$\sum_{\text{cyclic}} \frac{c(a^2+b^2)}{\sqrt{a^3+b^3}} \le \sum_{\text{cyclic}} c\sqrt{a+b}. \tag{2}$$

By the AM-GM Inequality, we have

$$c^2 + \frac{a+b}{2} \ge 2\sqrt{c^2\left(\frac{a+b}{2}\right)} = \sqrt{2}c\sqrt{a+b},$$

so

$$a^2+b^2+c^2+a+b+c = \sum_{\text{cyclic}}\left(c^2+\frac{a+b}{2}\right) \ge \sqrt{2}\sum_{\text{cyclic}} c\sqrt{a+b} \tag{3}$$

or, equivalently,

$$\sum_{\text{cyclic}} c\sqrt{a+b} \le \frac{1}{\sqrt{2}}(a^2+b^2+c^2+a+b+c). \tag{4}$$

From (2) and (4), our claim (1) follows immediately. It is easy to see that equality holds if and only if $a=b=c=1$.

Editor's comment. The stronger inequality featured above was also obtained by both Malikić and Perfetti. In addition, the following stronger inequality was obtained by Geupel :

$$\sum_{\text{cyclic}} \frac{c(a^2+b^2)}{\sqrt{a^3+b^3}} \le \frac{2}{3}(a^2+b^2+c^2)+\frac{3}{4}(a+b+c).$$

3775. [2012 : 334, 336] *Proposed by M. Chirita.*

Let $ABCD$ be a quadrilateral with $AC \perp BD$. Show that $ABCD$ is cyclic if and only if $BC \cdot AD = IA \cdot IB + IC \cdot ID$, where I is the point of intersection of the diagonals.

Solved by A. Alt; G. Apostolopoulos; Š. Arslanagić (2 solutions); M. Bataille; O. Kouba; D. Koukakis; S. Malikić; M.R.Modak; C. Sánchez-Rubio; Skidmore College Problem Group; D. Smith; I. Uchiha; D. Văcaru; P. Y. Woo; T. Zvonaru; and the proposer. We present 2 solutions.